Answermodel Test 2 2021-2022

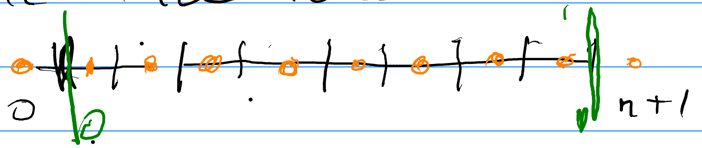
Exc 1 Consider the differential equation

$$\frac{du}{dx} - \frac{d}{dx}\left(e^{x} \frac{du}{dx}\right) = 0 \quad \text{on } (0,1]$$

with $u(0) = 1$ and $\frac{du}{dx}(1) = 2$

a Make a finite volume discretization of this equation where the boundary conditions coincide with a volume interface

$\boxed{1.7}$



° gridpoints

| volume interfaces.

$h = 1/n$

0.6 $$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{du}{dx} - \frac{d}{dx} \sim \, dx = 0$$

* $$u(x_{j+\frac{1}{2}}) - u(x_{j-\frac{1}{2}}) - e^{x_{j+\frac{1}{2}}}\left(\frac{du}{dx}\right)(x_{j+\frac{1}{2}}) + e^{x_{j-\frac{1}{2}}} \frac{du}{dx}(x_{j-\frac{1}{2}}) = 0$$

0.6 Discretization of $$u(x_{j+\frac{1}{2}}) - e^{x_{j+\frac{1}{2}}} \frac{du}{dx}(x_{j+\frac{1}{2}}) \simeq \frac{u(x_{j+1}) + u(x_j)}{2} - e^{x_{j+\frac{1}{2}}} \frac{u(x_{j+1}) - u(x_j)}{h} \equiv F_{j+\frac{1}{2}}$$

**

for $j = 0, \ldots, n$

0.5 $\quad \dot{u}_0 + u_1 = 2 \quad , \quad \frac{u_{n+1} - u_n}{h} = 2$

b Math: Show that the associated bilinear form has the shape .....

0.9

Start from form where bc's are eliminate.

Hence for $j=0$. we get $F_{\frac{1}{2}} = 0 - 1\,{}^{2u_1}/h$

$\int v\left(\frac{du}{dx} + \frac{d}{dx} \quad \right)dx$ $\qquad F_{n+\frac{1}{2}} = u_n$

$$a(\vec{v},\vec{u}) = \sum_{j=1}^{n} v_j\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) = \sum_{j=1}^{n} v_j F_{j+\frac{1}{2}} - \sum_{j=1}^{n} v_j F_{j-\frac{1}{2}}$$

0.3

$$= \sum_{j=1}^{n} \quad - \sum_{j=0}^{n-1} v_{j+1} F_{j+\frac{1}{2}} = -\sum_{j=1}^{n-1}\left(v_{j+1} - v_j\right)F_{j+\frac{1}{2}}$$

0.3

$$- v_1 F_{\frac{1}{2}} + v_n F_{n+\frac{1}{2}}$$

$$= \sum_{j=1}^{n-1}\left(v_{j+1} - v_j\right)\left(u_{j+1} + u_j\right)/2 + \left(v_{j+1} - v_j\right)e^{x_{j+\frac{1}{2}}}\left(u_{j+1} - u_j\right)/h$$

0.3

$$+ 2v_1 u_1/h + v_n u_n$$

C Show that $a(\vec{u},\vec{u})$ is nonnegative.

0.0

$$a(\vec{u},\vec{u}) = \sum_{j=1}^{n-1} -(u_{j+1}-u_j)(u_{j+1}+u_j)/2 + \underbrace{(u_{j+1}-u_j)^2 e^{x_{j+\frac{1}{2}}}}_{\geq 0}$$

$$+ 2u_1^2/h + u_n^2$$

0.2

$$= \sum_{j=1}^{n-1} -(u_{j+1}^2 - u_j^2)/2 + \quad / \quad + \sim$$

$$= -\frac{1}{2}\sum_{j=1}^{n-1} u_{j+1}^2 + \frac{1}{2}\sum_{j=1}^{n-1} u_j^2 + \ldots$$

$$= -\frac{1}{2}\sum_{j=2}^{n} u_j^2 + \ldots$$

0.2

$$= -\frac{1}{2}u_n^2 + \frac{1}{2}u_1^2 + \sim + 2u_1^2/h + u_n^2 =$$

$$= \left(\frac{1}{2}+\frac{2}{h}\right)u_1^2 + \frac{1}{2}u_n^2 + \sim$$

0.2

$$\geq 0$$

even $\sim$ only zero if u constant.
but then the other terms are positive

So form pos. def.

0.2

b ME Show that the discretization is monotonous for sufficient small h and give the criterion

0.8

Answer: Using * and ** we have

$$\frac{u_{j+1} - u_{j-1}}{(u_{j+1}-u_j) + (u_j - u_{j-1})} - e^{x_{j+\frac{1}{2}}}(u_{j+1}-u_j)/h + e^{x_{j-\frac{1}{2}}}(u_j - u_{j-1})/h = 0$$

0.5

$$\left(1 - e^{x_{j+\frac{1}{2}}}/h\right)(u_{j+1} - u_j) + \left(1 + e^{x_{j-\frac{1}{2}}}/h\right)(u_j - u_{j-1}) = 0$$

> 0

$$(u_{j+1} - u_j) = -\frac{(\quad)}{(\quad)}(u_j - u_{j-1})$$

0.2 monotonous if $h - e^{x_{j+\frac{1}{2}}} < 0$

$$h < e^{x_{j+\frac{1}{2}}}$$

0.1 Since we are on $[0,1]$ h will be less than 1 so h allways be less than $e^{x_{j+\frac{1}{2}}}$

CME:

$\boxed{0{,}9}$ Discretization

$$\frac{u_{j+1}-u_{j-1}}{2h} + \left(e^{x_{j+\frac12}}(u_{j+1}-u_j) - e^{x_{j-\frac12}}(u_j-u_{j-1})\right)/h^2 = 0$$

0.3

$$u_{j+1} = u_j + h u'(x_j) + \frac{h^2}{2} u''(x_j) + \frac{h^3}{6} u'''(x_j) + \frac{h^4}{24} u^{IV}(\xi)$$

$$\frac{u_{j+1}-u_{j-1}}{2h} = u'(x_j) + \frac{h^2}{12} u'''(x_j) + O(h^3)$$

second term

$$e^{x_{j+\frac12}}(u_{j+1}-u_j) - e^{x_{j-\frac12}}(u_j-u_{j-1}) =$$

$$= e^{x_{j+\frac12}}\left(h u_j' + \frac{h^2}{2} u_j'' + \frac{h^3}{6} u_j''' + O(h^4)\right) -$$

$$e^{x_{j-\frac12}}\left(h u_j' - \frac{h^2}{2} u_j'' + \frac{h^3}{6} u''' + O(h^4)\right)$$

0.2

$$= h\left(e^{x_{j+\frac12}} - e^{x_{j-\frac12}}\right) u_j' + \frac{h^2}{2}\left(e^{x_{j+\frac12}} + e^{x_{j-\frac12}}\right) u_j''$$

$$+ \frac{h^3}{6}\left(e^{x_{j+\frac12}} - e^{x_{j-\frac12}}\right) u_j''' + O(h^4) =$$

0.2

Intermezzo: $e^{x_{j+\frac{1}{2}}} = e^{x_j} + \frac{1}{2} h e^{x_j} + \frac{1}{8} h^2 e^{x_j} + O(h^3)$

$$\left(e^{x_{j+\frac{1}{2}}} - e^{x_{j-\frac{1}{2}}}\right) = h e^{x_j} + O(h^3)$$

$$\left(e^{x_{j+\frac{1}{2}}} + e^{x_{j-\frac{1}{2}}}\right) = 2 e^{x_j} + O(h^2)$$

Hence:

$$= h^2 e^{x_j} u_j' + O(h^4) + h^2 e^{x_j} u_j'' + O(h^4) + O(h^4)$$

0.2

$$= h^2 \left(e^{x_j} u_j' + e^{x_j} u_j'' \right) + O(h^4)$$

$$= h^2 \left(\frac{d}{dx} e^x \frac{du}{dx} \right)_{x=x_j} + O(h^4)$$

Exc 2 Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \quad \text{on } [0,1] \times [0,1]$$

$$u(0,y,t) = f(t), \quad \frac{\partial u}{\partial x}(1,y,t) = 0$$

$$\frac{\partial u}{\partial y}(x,0,t) = \frac{\partial u}{\partial y}(x,1,t) = 0$$

a Transform this equation into a system of first-order ODEs by a finite difference discretization.

[1.]

The Dirichlet condition is at the grid point.
The Neumann conditions in the middle between 2 grid points

Use subscripts j, k to indicate a gridpoint (x_j, y_k)

Answer:

aisr

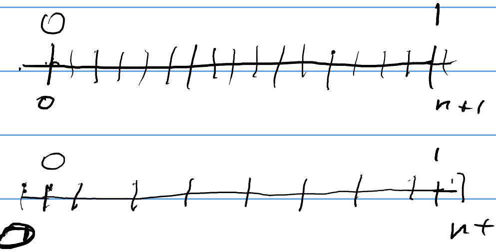
0.5 $$\frac{d^2}{dt^2} u_{jk} = c^2 \{ (u_{j+1,k} - 2u_{jk} + u_{j-1,k}) / \Delta x^2 +$$

$$\left(u_{j,k+1} - 2u_{jk} + u_{j,k-1}\right)/\Delta y^2 \Big\}$$

for $j = 1, \dots, n$
$k = 1, \dots, n$

grid

0.3



$\Delta x = 1/(n+\frac{1}{2})$

$\Delta y = 1/n$

bc's

0.3

$u_{0,k} = f(t)$, $u_{n+1,k} - u_{n,k} = 0$, $u_{j,1} - u_{j,0} = 0$, $u_{j,n+1} - u_{j,n} = 0$

$u_{jk}(0) = 0$, $\frac{du_{jk}}{dt}(0) = 0$ $j, k = 1, \dots, n$

subst boundary conditions

0.3

$$\frac{d^2}{dt^2} u_{1k} = c^2 \left\{ \frac{u_{2k} - 2u_{1k}}{h^2} + \frac{u_{1,k+1} - 2u_{1k} + u_{2,1}}{k^2} \right\} + \frac{c^2 f(t)}{h^2}$$

$$\frac{d^2}{dt^2}$$

goesto
b(t)

$$\vec{u} = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,n} \\ u_{2,1} \\ \vdots \\ u_{2,n} \\ \vdots \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

0,3

$$A = c^2 \begin{bmatrix} \ddots & & \ddots & & \ddots & & \ddots & \\ \frac{1}{k^2} & \frac{1}{h^2} & -\frac{2}{h^2} - \frac{2}{k^2} & + \frac{1}{h^2} & + \frac{1}{k^2} \\ & \ddots & & \ddots & & \ddots & & \ddots \end{bmatrix}$$

b $\boxed{1.2}$ A 2 dimensional Fourier component is given by $e^{i(j\theta + k\varphi)}$

Determine Fourier eigenvalues from A

Answer

Consider first x-direction $u_{j,k} = e^{i(j\theta + k\varphi)}$

0.5 $(u_{j+1,k} - 2u_{j,k} + u_{j-1,k})/\Delta x^2 =$

$\left(e^{i[(j+1)\theta + k\varphi]} - 2e^{i(j\theta + k\varphi)} + e^{i((j-1)\theta + k\varphi)}\right)/\Delta x^2$

0.1 $= (e^{i\theta} - 2 + e^{-i\theta})/\Delta x^2 \; e^{i(j\theta + k\varphi)}$

0.2 $= -2(1 - \cos\theta)\, e^{i(j\theta + k\varphi)}$

0.1 $= \dfrac{-4 \sin^2 \theta/2}{\Delta x^2} e^{i(j\theta + k\varphi)}$

similar for y-direction

0.3 $\lambda(\theta, \varphi) = -\dfrac{4 \sin^2 \theta/2}{\Delta x^2} - \dfrac{4 \sin^2 \varphi/2}{\Delta x^2}$

C $\boxed{0.8}$ First-order system

$$\frac{du_{jk}}{dt} = v_{jk},$$

$$\frac{dv_{jk}}{dt} = c^2\{(u_{j+1,k} - \cdots\}$$

b.c. as before

$u_{jk}(0) = 0$, $v_{jk}(0) = 0$ for $j,k = 1,2,\cdots,n$

or

0.8

$$\frac{d\vec{u}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = A\vec{u} + b(t)$$

d

0.3

⊗ Use this result to compute the Fourier eigenvalues of the r.h.s of the first order equation.

Answer: replace

$$\det\left(\begin{bmatrix} 0 & 1 \\ \lambda(\theta,\varphi) & 0 \end{bmatrix} - \mu\right) = 0 \quad \text{gives}$$

$$\rightarrow \mu^2 = \lambda(\theta,\varphi)$$

$$\mu = \pm\sqrt{\lambda(\theta,\varphi)}$$

So we have purely imaginary eigenvalues for the first order system.

0.3

$$\mu(\theta,\varphi) = \pm 2i\sqrt{\frac{\sin^2\theta/2}{\Delta x^2} + \frac{\sin^2\varphi/2}{\Delta y^2}}$$

e) The classical Runge-Kutta 4 method

0.8

contains in its abs. stability domain the part $[-2.7i, 2.7i]$ on the imaginary axis.

Determine the max. time step one can take with this method on the first order system.

Answer: We see that $\lambda(\theta,\varphi) \in \left[-\frac{8}{\Delta x^2} - \frac{4}{\Delta y^2}, 0\right]$

Or $\mu(\theta,\varphi) \in \left[-i2\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}},\ i2\sqrt{\quad}\right]$ 0.3

Condition

$\Delta t\, \mu(\theta,\varphi) \in [-2.7i, 2.7i]$ 0.3

$\Rightarrow \Delta t\, 2\sqrt{\quad} < 2.7$ 0.2

$$\Delta t = \frac{2.7}{2}\frac{1}{\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}}}$$

$\boxed{0.5}$ Given the eigenvalue problem resulting . the second-order system. for computing the eigen frequencies.

$$\vec{u} = \vec{v} e^{i\omega t} \quad \text{or} \quad \vec{u} = \vec{v} e^{i 2\pi f t}$$

$$-\omega^2 \cancel{e^{i\omega t}} v = \cancel{e^{i\omega t}} \left\{ \frac{v_{j+1,k} - 2v_{j,k} + \cdots}{\Delta x^2} + v_{j,k+1} - 2v_{j,k} \right\}$$

0,5 $$\underbrace{-(2\pi f)^2}_{\lambda} v = A v$$

$$+ \text{bc} \quad v_{n+1,k} - v_{n,k} = 0, \quad v_{0,k} = 0$$

$$(v_{j,n+1} - v_{j,n}) = 0, \quad v_{j,1} - v_{j,0} = 0$$

We can solve the eigenvalue problem

$$A v - \lambda v = 0$$

then $\omega^2 = -\lambda \quad \omega = \pm\sqrt{-\lambda} \rightarrow f = \frac{\omega}{2\pi}$

$$\rightarrow f = \pm \frac{1}{2\pi}\sqrt{-\lambda}$$

9 Show that the power method converges to the eigenvector corresponding to the largest eigenvalue and also give the convergence rate

Answer:
Power method is essentially

$$x_k = A^k x_0 .$$

Set

$$x_0 = \sum_{i=1}^{n} \alpha_i v_i , \quad A v_i = \lambda_i v_i \quad i = 1, \dots, n$$

Suppose $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \dots |\lambda_n|$

$$x_k = A^k \sum \dots = \sum \alpha_i A^k v_i = \sum \alpha_i \lambda_i^k v_i$$

$$= \alpha_1 \lambda_1^k v_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^k v_i =$$

$$= \lambda_1^k \left[\alpha_1 v_1 + \sum \alpha_i \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i \right] \cong \lambda_1^k \alpha_1 v_1$$

scalar eigenvector

for k large

$|\cdot| < 1$ goes to zero for $k \to \infty$

9 [redacted] How can one use the power method to find the frequency nearest to a target frequency f_0

$\boxed{0.6}$

Target for $\lambda_0 = -\omega_0^2 = -\frac{f_0^2}{(2\pi)^2}$

Replace A by $(A - \lambda_0 I)^{-1}$ ← because the eigen value nearest to λ_0 will become the largest one

0.3

~~Fin~~ One finds an eigenvalue μ_1

Then $\frac{1}{(\lambda_1 - \lambda_0)} = \mu_1$ 0.1

$\rightarrow \lambda_1 - \lambda_0 = \frac{1}{\mu_1} \rightarrow \lambda_1 = \lambda_0 + \frac{1}{\mu_1}$

0.1

$\omega_1 = \sqrt{-\lambda_1}$

$f_1 = \sqrt{-\lambda_1}/2\pi = \sqrt{-(\lambda_0 + \frac{1}{\mu_1})}/2\pi$

$\boxed{0.1}$